

Generalized Parton Distributions for large x

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(Dated: February 1, 2008)

The t -dependence of generalized parton distributions for $x \rightarrow 1$ is discussed. We argue that constituent quark models, where the t -dependence for $x \rightarrow 1$ is through the product $(1-x)t$, are inconsistent. Instead we suggest a leading dependence in terms of $(1-x)^n t$, where $n \geq 2$, for $x \rightarrow 1$.

PACS numbers:

I. INTRODUCTION

Generalized parton distributions (GPDs) [1, 2, 3] are a very powerful theoretical tool which allows linking parton distributions with form factors as well as many other hadronic matrix elements (for a recent review, see Ref. [4]). Unfortunately, they cannot be measured directly but instead they appear in convolution integrals of the form

$$\text{Amplitude}(\xi, t) \sim \int dx \frac{GPD(x, \xi, t)}{x - \xi \pm i\varepsilon}. \quad (1)$$

Since these convolution integrals cannot be easily inverted, GPDs are often ‘extracted’ from the data by writing down a model ansatz with various free parameters which are then fitted to the data. In order to reduce the arbitrariness in this procedure, it is important to incorporate as many theoretical constraints as possible into the ansatz. For the intermediate and large x region a commonly used ansatz for GPDs starts from a simple model for light-cone wave functions. For example, for the case of the pion one writes down a 2-particle wave function of the form

$$\psi(x, \mathbf{k}_\perp) \sim f(x) \exp(-\text{const.} \mathcal{M}), \quad (2)$$

where one conveniently chooses \mathcal{M} such that wave function components with a high kinetic energy are suppressed

$$\mathcal{M} = \frac{m^2 + \mathbf{k}_\perp^2}{x} + \frac{m^2 + \mathbf{k}_\perp^2}{1-x}. \quad (3)$$

Upon inserting this type of ansatz into the convolution equations for GPDs [5] at $\xi = 0$

$$H(x, 0, t) = \int d^2 \mathbf{k}_\perp \psi^*(x, \mathbf{k}_\perp) \psi(x, \mathbf{k}_\perp + (1-x) \mathbf{q}_\perp), \quad (4)$$

one finds [3] a t -dependence ($t \equiv -\mathbf{q}_\perp^2$) that is suppressed by one power of $(1-x)$ for $x \rightarrow 1$

$$H(x, 0, t) = q(x) \exp\left(at \frac{1-x}{x}\right). \quad (5)$$

Generalizations of Eqs. (2) and (3) to more than two constituents (higher Fock components, baryon) yield the same kind of t dependence as Eq. (5).

Obviously, Eq. (5) gives rise to the wrong behavior for $x \rightarrow 0$ (transverse size grows like $\frac{1}{\sqrt{x}}$), but this does not come as a surprise since one would not expect a good description at small x from a valence model. Moreover, when $Q^2 > \text{a few } GeV^2$ then the small x behavior of GPDs is practically irrelevant for form factors and Compton scattering. Therefore, we will not concern ourselves here with the flaws of the above ansatz at small x .

However, it is widely believed [3, 6] that Eq. (5) provides a qualitatively reasonable description in the region of intermediate and large x , where a constituent model for hadrons has a chance to make sense.

In this letter, we argue that the $x \rightarrow 1$ behavior of Eqs. (2-5) is inconsistent.

II. TRANSVERSE SIZE

Upon Fourier transforming Eq. (5) to impact parameter space [7, 8, 9, 10, 11], one finds

$$\begin{aligned} q(x, \mathbf{b}_\perp) &= \int \frac{d^2 \mathbf{q}_\perp}{(2\pi)^2} e^{i \mathbf{b}_\perp \cdot \mathbf{q}_\perp} H(x, 0, -\mathbf{q}_\perp^2) \\ &= q(x) \frac{1}{4\pi a} \frac{x}{1-x} \exp\left(-\frac{\mathbf{b}_\perp^2}{4a} \frac{x}{1-x}\right), \end{aligned} \quad (6)$$

i.e. the width of this distribution in impact parameter space behaves like

$$\langle \mathbf{b}_\perp^2 \rangle \equiv \frac{\int d^2 \mathbf{b}_\perp \mathbf{b}_\perp^2 q(x, \mathbf{b}_\perp)}{\int d^2 \mathbf{b}_\perp q(x, \mathbf{b}_\perp)} \propto 1-x \quad (7)$$

as $x \rightarrow 1$. However, \mathbf{b}_\perp only measures the distance from the active quark to the center of momentum of the hadron. A better measure for the size (diameter) of a configuration of the wave function is given by the separation \mathbf{r}_\perp between the active quark and the center of momentum of the spectators (actually even \mathbf{r}_\perp is only a lower bound on the diameter). In terms of \mathbf{b}_\perp one finds for the separation $\mathbf{r}_\perp = \frac{\mathbf{b}_\perp}{1-x}$. With the above ansatz (5) the transverse size diverges as $x \rightarrow 1$

$$\langle \mathbf{r}_\perp^2 \rangle \equiv \frac{\int d^2 \mathbf{b}_\perp \frac{\mathbf{b}_\perp^2}{(1-x)^2} q(x, \mathbf{b}_\perp)}{\int d^2 \mathbf{b}_\perp q(x, \mathbf{b}_\perp)} \sim \frac{1}{1-x}. \quad (8)$$

For this result it was irrelevant that the t -dependence in Eq. (5) is exponential. The important feature that led

to Eq. (8) was the fact that the dependence on t was through the combination $(1-x)t$ for $x \rightarrow 1$.

This power law growth of the transverse size as $x \rightarrow 1$ is not only bizarre, but in fact it makes the logic behind the valence ansatz (2), (3) for the light-cone wave function inconsistent: First of all, if a $q\bar{q}$ pair is separated by a large \perp distance then its potential energy is very high and therefore one cannot neglect the potential energy in Eq. (3). In the next section we will provide an estimate of that potential energy. Secondly, a $q\bar{q}$ pair that has a \perp separation must be connected by a \perp gauge string — otherwise its energy is infrared divergent. Even if one does not put in such a gauge string “by hand”, any non-perturbative diagonalization of a light-cone Hamiltonian for QCD should yield such a string in the wave function for finite energy hadrons. The presence of such a \perp gauge string automatically implies that this component of the wave function also contains gluon degrees of freedom in contradiction with the valence ansatz that was used as a starting point (2) and (3). In fact, since the \perp separation between the quark and the antiquark diverges as $x \rightarrow 1$, such a state would have to contain an infinite number of gluons as $x \rightarrow 1$. Similar reasoning applies to a nucleon valence ansatz analogous to Eq. (3).

In summary, the whole logic that starts from a valence ansatz for the light-cone wave function, in which the \perp momentum dependence is only governed by the kinetic energy, becomes inconsistent in the limit $x \rightarrow 1$. The above light-cone wave function model is the only justification for writing down a t -dependence of the form $\exp(at\frac{1-x}{x})$ for large x . Hence we are led to abandon Eq. (5) for large x . Of course, for intermediate x , Eq. (5) may still provide a reasonable and consistent description. However, since form factors and Compton amplitudes, are rather sensitive to the behavior of $H(x, 0, Q^2)$ for $x > 0.5$ when $Q^2 > 10\text{GeV}^2$, it becomes necessary to improve on the $x \rightarrow 1$ behavior of Eq. (5).

III. AN IMPROVED ANSATZ FOR $H(x, 0, t)$

If one really wants to know $H(x, 0, t)$ for $x \rightarrow 1$, all one has to do is solve QCD. Since we are not yet ready to perform such calculations with the required accuracy, we want to propose in this section an improved model ansatz for $H(x, 0, t)$ for $x \rightarrow 1$.

Even without doing any calculation, it is clear that in order to cure the problem of increasing size as $x \rightarrow 1$, the t -dependence must be suppressed with a higher power of $(1-x)$: a finite \perp size as $x \rightarrow 1$ is achieved if and only if the dependence on t for $x \rightarrow 1$ is of the form $t(1-x)^n$ with $n \geq 2$. In this section, we would like to present a variational argument to provide a more precise estimate for the $x \rightarrow 1$ behavior.

For this purpose, let us estimate the potential energy contribution to the light-cone Hamiltonian for a $q\bar{q}$ pair that is separated by a \perp distance \mathbf{r}_\perp . Assuming a linear potential, the effective mass of the QCD string connect-

ing the $q\bar{q}$ pair is at least

$$m_g = \sigma |\mathbf{r}_\perp|. \quad (9)$$

The quantity that enters the light-cone Hamiltonian is the invariant mass of the glue divided by the light-cone momentum carried by the glue. Without actually solving (i.e. diagonalizing) the light-cone Hamiltonian one cannot know how the light-cone momentum is divided among the antiquark and the string of glue, but obviously the glue cannot carry more than momentum fraction $1-x$ if x is the momentum fraction carried by the active quark. This motivates us to consider the following (conservative) ansatz for the light-cone energy of the $q\bar{q}$ pair including the effects of the potential energy at large separations

$$\tilde{\mathcal{M}} = \mathcal{M} + \frac{\sigma^2 \mathbf{r}_\perp^2}{1-x}, \quad (10)$$

where $\sigma \approx (440 \text{ MeV})^2$ is the string tension. This ansatz is conservative because, as we explained above, it only underestimates the light-cone energy of the glue. Nevertheless, let us estimate the effect of adding such a term in the effective Hamiltonian for a constituent model of the pion.

For $x \rightarrow 1$, the variables \mathbf{k}_\perp^2 and \mathbf{r}_\perp^2 (which are Fourier conjugate to each other) appear symmetrically in $\tilde{\mathcal{M}}$ (that is up to the factor $c\sigma^2$, which provides the scale): both are divided by one power of $(1-x)$. Therefore if one considers in the light-cone energy an ansatz that includes the transverse gauge string tension (in this ansatz we are only concerned about the singularity in the energy for $x \rightarrow 1$)

$$\tilde{\mathcal{M}} = \frac{m^2}{x(1-x)} + \frac{\mathbf{k}_\perp^2}{x(1-x)} + \frac{\sigma^2 \mathbf{r}_\perp^2}{1-x} \quad (11)$$

one expects that the \mathbf{k}_\perp -dependence of the light-cone wave function factorizes near $x \rightarrow 1$, i.e.

$$\psi(x, \mathbf{k}_\perp) \xrightarrow{x \rightarrow 1} (1-x)^k \phi(\mathbf{k}_\perp). \quad (12)$$

Inserting this result into the convolution formula (4) yields GPDs where the t -dependence is suppressed by 2 powers of $(1-x)$ near $x \rightarrow 1$.

One may criticize that instead of solving the light-cone Hamiltonian for QCD, we only performed some dimensional analysis. However, there is certainly no justification to suppress in an ansatz of the light-cone wave function only components that have a high kinetic energy but not those that have a high potential energy (2) — especially if the resulting state yields a potential energy that diverges badly as $x \rightarrow 1$. In contrast, our ansatz at least builds in some of the effects from the gluon field and thus gives rise to a finite size for $x \rightarrow 1$, and potential and kinetic energy scale in the same way for $x \rightarrow 1$. A finite size in position space suggests a finite size in momentum space. Since the convolution expressions for GPDs involve the \perp momentum transfer with a factor of $(1-x)$

this naturally leads to GPDs where the t -dependence for $x \rightarrow 1$ should be through the combination $t(1-x)^2$, i.e. a better ansatz to describe the $x \rightarrow 1$ behaviour of GPDs reads

$$H(x, 0, t) \xrightarrow{x \rightarrow 1} q(x) \exp[at(1-x)^2]. \quad (13)$$

However, we neglected many things in our analysis. For example, the ansatz for the potential energy term included on the \perp string tension and we potentially even underestimated that contribution, which leaves open the possibility that the suppression of the t -dependence may be of the form $t(1-x)^n$ with $n \geq 2$. Although our analysis clearly rules out $n < 2$, it is too crude to specify the correct value of n uniquely.

Of course, it is quite possible that the actual behavior of GPDs at large x is even more complicated than the semi-factorized form in Eq. (13). Because of this possibility, one should regard Eq. (13) only as one possibility to illustrate the difference to the previously used ansatz. Nevertheless, our main point, i.e. the fact that the t -dependence of GPDs for $x \rightarrow 1$ should be suppressed by at least 2 powers of $(1-x)$ should be model-independent. This general result is also supported by perturbative QCD [12] studies as well as recent lattice gauge theory results [13]. In Ref. [12] it was found that in pQCD the t -dependent terms near $x \rightarrow 1$ are suppressed by an additional power of $(1-x)^2$ near $x \rightarrow 1$. Lattice gauge theory calculations of the r.m.s. radii for the lowest moments of $H(x, 0, t)$ indicate a strong suppression for the r.m.s. radii of subsequent moments, which indicates a suppression of the t dependence with $(1-x)^n$ near $x \rightarrow 1$ where $n > 1$. Recent transverse lattice calculations [14] even indicate a shrinking \perp size as $x \rightarrow 1$, i.e. $n > 2$, in the case of the pion.

It is interesting to note that GPDs, which have a t -dependence that comes with a factor of $(1-x)^2$ near $x \rightarrow 1$, naturally give rise to Drell-Yan-West duality between parton distributions at large x and the form factor. Indeed, the ansatz

$$H(x, 0, t) = (1-x)^{2N_s-1} \exp[at(1-x)^2] \quad (14)$$

yields

$$F(t) = \int dx H(x, 0, t) \quad (15)$$

$$\xrightarrow{t \rightarrow -\infty} \frac{\Gamma(N_s)}{2a^{N_s}} \frac{1}{(-t)^{N_s}}. \quad (16)$$

Here only the behavior near $x \rightarrow 1$ matters and DYW duality would also arise (with a different coefficient) if the exponential function were being replaced by any function that falls rapidly for $t \rightarrow -\infty$, x fixed.

We should also point out that there is an interesting connection between the value of n and the occurrence of

color transparency [15], where color transparency does not occur for $n < 2$.

IV. SUMMARY

We have demonstrated that a commonly employed ansatz for light-cone wave functions, where the \mathbf{k}_\perp -dependence is only through the light-cone kinetic energy of the constituents, is inconsistent for $x \rightarrow 1$ because it leads to a divergent transverse size for those wave function components where one constituent carries $x \rightarrow 1$. If the constituents are separated by a divergent \perp distances then they must be connected by \perp strings of color flux of infinite length, which in the infinite momentum frame correspond to an infinite number of transverse gluons. Therefore such a configuration corresponds to a very high Fock component and *not* to a valence component. Hence the initial valence ansatz becomes inconsistent and there is no reason to trust the $x \rightarrow 1$ behavior of GPDs that are based on such a valence ansatz (with \mathbf{k}_\perp dependence in the wave function only through the light-cone kinetic energy). Since such an ansatz has been the only motivation for parameterizing GPDs with function where the leading t dependence for $x \rightarrow 1$ is through the product $(1-x)t$, we are led to abandon such parameterizations for GPDs.

The origin for this divergent size problem is the fact that the distance between the active quark and the center of momentum of the spectators $\mathbf{r} \equiv \mathbf{r}_{\perp\mathbf{q}} - \mathbf{r}_{\perp\bar{\mathbf{q}}}$ is related to the impact parameter \mathbf{b}_\perp (the distance from the active quark to the center of momentum) via $\mathbf{r}_\perp = \mathbf{b}_\perp/(1-x)$. In order for the hadron to have a finite size as $x \rightarrow 1$, one must have $\langle \mathbf{b}_\perp^2 \rangle \sim (1-x)^n$ with $n \geq 2$, which in turn requires that the leading dependence on t is through the product $(1-x)^n t$ with $n \geq 2$.

While we are not able to predict what the actual behavior is for $x \rightarrow 1$, we made an attempt to include the gluonic energy into an ansatz for the light-cone wave function. With such an ansatz, large size configurations are naturally suppressed and we find GPDs where the $x \rightarrow 1$ dependence on t is through the combination $(1-x)^n t$ with $n = 2$. Of course the estimate that led to this result was very crude and therefore one should not take the value $n = 2$ as a rigorous prediction. Nevertheless, we believe that $n \geq 2$ is a much more reasonable choice in parameterizations than $n = 1$ since $n = 2$ is consistent with hadrons that have a finite size (actually for $n > 2$ it would even be consistent with a vanishing size) for $x \rightarrow 1$.

Another interesting observation concerns the Drell-Yan-West duality between the form factor and structure functions since the case $n = 2$ naturally leads to the same duality relation as quark counting rules.

Acknowledgements: I would like to thank G. Miller and D. Renner for stimulating discussions. This work was supported by the DOE under grant number DE-FG03-95ER40965.

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